NIDS Session 7

$\boxed{1}$ Let's consider

$$\begin{cases} y'(t) = f(y(t)) \\ y(0) = y_0 \end{cases}$$ which ρ-reversible

meaning $\rho \circ f = - f \circ \rho$

consider a RK method:

$$k_i = f\left(y_0 + h \sum_{j=1}^{s} a_{ij} k_j\right)$$

$$y_1 = y_0 + h \sum_{i=1}^{s} b_i k_i$$

Goal: show that $\rho \circ \varphi_h = \varphi_{-h} \circ \rho$

where φ_h is the numerical flow

Let $y_0 \in \mathbb{R}^d$, we compute $(\rho \circ \varphi_h)(y_0)$

$$\rho(\varphi_h(y_0)) = \rho(y_1)$$

$$= \rho(y_0) + h \sum_{i=1}^{s} b_i \rho(k_i) \qquad (1)$$

(by the linearity of ρ)

Let's define $\tilde{k}_i = -p(k_i)$

$$\tilde{k}_i = -p\Big(f\Big(y_0 + h\sum_{j=1}^{s} a_{ij} k_j\Big)\Big)$$

$$= f\Big(p(y_0) + h\sum_{j=1}^{s} a_{ij}\, p(k_j)\Big)$$

$$= f\Big(p(y_0) - h\sum_{j=1}^{s} a_{ij} \tilde{k}_j\Big) \qquad (2)'$$

(1) becomes

$$p(\Phi_h(y_0)) = p(y_0) - h\sum_{i=1}^{s} b_i \tilde{k}_i \qquad (1)'$$

(1)' and (2)' define an RK method with y_0 substituted with $p(y_0)$ and h replaced with $-h$

Hence

$$p(\Phi_h(y_0)) = \Phi_{-h}(p(y_0))$$

2. Again consider the same problem

$$\begin{cases} y'(t) = f(y(t)) \\ y(0) = y_0 \end{cases}$$

We want to show that $\varphi_t \circ \varphi_s(y_0)$

$$= \varphi_{t+s}(y_0)$$

Let us define

$$y_1(\tau) = y(\tau, 0, y(s, 0, y_0))$$

$$= y(\tau, 0, \varphi_s(y_0))$$

$$= \varphi_\tau(\varphi_s(y_0))$$

$$y_2(\tau) = y(\tau + s, 0, y_0)$$

$$= \varphi_{\tau+s}(y_0)$$

Note that $y_1(\tau)$ solves

$$\begin{cases} y_1'(\tau) = f(y_1(\tau)) \\ y_1(0) = \varphi_0(\varphi_s(y_0)) = y(s) \end{cases}$$

and $y_2(\tau)$ solves

$$y_2'(\tau) = f(y_2(\tau))$$

$$y_2(0) = \varphi_s(y_0) = y(s)$$

Both $y_1(\tau)$ and $y_2(\tau)$ solve the same differential problem with the same initial conditions. By the Cauchy - Lipschitz theorem, there exists a unique maximal solution, thus $y_1(\tau) = y_2(\tau)$

$$\varphi_\tau(\varphi_s(y)) = \varphi_{\tau+s}(y_0)$$

$$\Leftrightarrow \quad \varphi_\tau \circ \varphi_s = \varphi_{\tau+s}$$

in particular $\varphi_h \circ \varphi_{-h} = \varphi_0 = id$

ii) By definition $\Phi_h^* = \Phi_{-h}^{-1}$

If the method is symmetric

$$\Phi_h \circ \Phi_{-h} = id$$

$$\Leftarrow \quad \Phi_n = \Phi_{-n}^{-1} = \Phi_n^*$$

(iii) $$(\Phi_n^*)^* = (\Phi_{-n}^{-1})^*$$
$$= (\Phi_n^{-1})^{-1} = \Phi_n$$

(iv) $$(\Phi_n \circ \Psi_n)^*$$
$$= (\Phi_{-n} \circ \Psi_{-n})^{-1}$$
$$= \Psi_{-n}^{-1} \circ \Phi_{-n}^{-1}$$
$$= \Psi_n^* \circ \Phi_n^*$$

$\boxed{3}$ Summary from point i) :

$$\begin{cases} b_i^* = b_{n+1-i} & (1) \\ a_{ij}^* = b_{n+1-j} - a_{n+1-i,\, n+1-j} & (2) \\ c_i^* = 1 - c_{n+1-i} & (3) \end{cases}$$

Assume that $a_{ij} = b_j - a_{n+1-i,\, n+1-j}$

$$\Leftrightarrow \quad a_{n+1-i,\, n+1-j} = b_{n+1-j} - a_{ij}$$

Substituting in (2), we obtain

$$a_{ij}^* = a_{ij}$$

Moreover,

$$c_i = \sum_{j=1}^{n} a_{ij} = \sum_{j=1}^{n} b_j - a_{n+1-i\, n+1-j}$$

$$= \underbrace{\sum_{j=1}^{n} b_j}_{=1} - \underbrace{\sum_{j=1}^{n} a_{n+1-i\, n+1-j}}_{= c_{n+1-i}}$$

$$= 1 - c_{n+1-i}$$

$$= c_i^* \quad (\text{by } (3))$$

Finally,

$$\begin{cases} a_{ij}^* = b_{s+1-j} - a_{s+1-i\,s+1-j} & (1)' \\ a_{ij} = b_j - a_{s+1-i\,s+1-j} & (2)' \end{cases}$$

(since $a_{ij}^* = a_{ij}$)

$(1)' - (2)' \Rightarrow \quad b_j = b_{s+1-j} = b_j^*$

(iii) from (2), with $j = i$,

$$a_{ii}^* = b_{s+1-i} - \underbrace{a_{s+1-i\,s+1-i}}_{=0}$$

$= 0$ because the method is explicit

$$a_{ii}^* = b_{s+1-i}$$

However, since $\sum_{i=1}^{s} b_i = 1$, there exists at least one index i such that

$$a_{ii}^* = b_{s+1-i} \neq 0 = a_{ii}$$

Thus, the method cannot be symmetric.

$\boxed{4}$ By assumption the RK method is irreducible, so the Cooper conditions are satisfied $\left(b_i b_j = b_i a_{ij} + b_j a_{ji} \right)$

$$b_i^* a_{ij}^* + b_j^* a_{ji}^*$$

$$= b_{s+1-i} \left(b_{s+1-j} - a_{s+1-i,\, s+1-j} \right)$$

$$+ b_{s+1-j} \left(b_{s+1-i} - a_{s+1-j,\, s+1-i} \right)$$

$$= 2\, b_{s+1-i} b_{s+1-j}$$

$$- \underbrace{\left(b_{s+1-i}\, a_{s+1-i,\, s+1-j} + b_{s+1-j}\, a_{s+1-j,\, s+1-i} \right)}_{= b_{s+1-i}\, b_{s+1-j}}$$

$$= b_{s+1-i}\, b_{s+1-j} = b_i^* b_j^*$$

Since the Cooper condition is satisfied for the adjoint, it conserves quadratic invariants